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UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

ON OPTIMAL HARVESTING WITH AN APPLICATION TO AGE-STRUCTURED POPULATIONS

Morton E. Gurtin* and Lea F. Murphy**

Technical Summary Report #2200

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ABSTRACT

This paper develops optimal harvesting policies for a population whose evolution is governed by a single autonomous nonlinear differential equation. The objective functional is not assumed to be convex. The results are used to discuss optimal policies for age-structured populations harvested with effort independent of age.

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Key Words: Population dynamics, harvesting

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SIGNIFICANCE AND EXPLANATION

This paper develops optimal policies for harvesting biological populations. The age structure of the population is taken into account. The policies generally involve harvesting at maximal effort, not harvesting, and harvesting to keep the growth rate of the population a maximum.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON OPTIMAL HARVESTING WITH AN APPLICATION TO
AGE-STRUCTURED POPULATIONS

Morton E. Gurtin* and Lea F. Murphy**

1. Introduction.

In this paper we develop optimal-harvesting policies for a population whose evolution is governed by a single autonomous nonlinear differential equation.[1] We present a formulation of the infinite-horizon problem which does not involve a discount rate[2] and, using straightforward arguments, are able to establish the following results:

(a) If the effort is unconstrained, the optimal policy is to reach a certain value P_* of total population as quickly as possible, and then to hold the population at that value for all subsequent time. As one would expect, P_* is the population size that maximizes the growth rate and hence corresponds to the maximum sustainable yield.

(b) This policy is also optimal when the effort is constrained to be less than a constant $\bar{E}$, as long as $\bar{E}$ is sufficiently large. For $\bar{E}$ small, however, it may not be possible to hold the population $P(t)$ constant at P_*. In this case, the optimal policy involves a "bang-bang" effort function and the total population approaches a limit larger than P_*.

We prove these results directly, without the aid of classical control theory. Indeed, the classical theory is not well suited to this problem, since we deal, in part, with unbounded controls, and since our formulation of the infinite horizon problem is nonstandard.

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[1] This problem has been given considerable attention in the literature (cf., e.g., Clark and Munro [1], Clark [2], Spence and Starrett [3]). Our results (b) for P_0 and $\bar{E}$ small appear to be new.

[2] We use the overtaking criterion of optimality.

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We use our results to discuss optimal policies for <u>age-structured</u> populations harvested with effort independent of age. Using a simplified form, due to Coleman and Simmes [4], of a general nonlinear model introduced by Gurtin and MacCamy [5], we are able to show that for a population whose age distribution is initially persistent[1] - an assumption justified for a population which has been evolving over a long period of time - the optimal-harvesting problem reduces to the problem discussed above.

2. Age-independent theory.

a. The optimal-harvesting problem.

We consider a species whose total population $P(t)$, when not harvested, is governed by a differential equation of the form

$$\dot{P} = y(P)$$

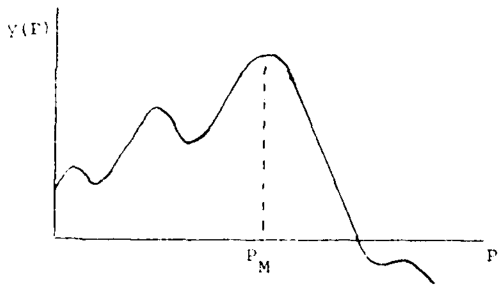

Figure 1. The growth rate y.

We assume that the growth rate y is a C^1 function on $[0,\infty)$ with a

strict global maximum at $P_M > 0$. (1)

We assume further that, on $(0,P_M)$, y is strictly-positive[2] and has a finite number of critical points.

---

[1] We use the term "persistent age distribution" for what in the literature is usually referred to as a "stable age distribution." In a future paper [6] we will investigate the consequences of arbitrary initial conditions.

[2] This assumption can be weakened to include "critical depensation." Indeed, we need only assume that, when $P_0 < P_M$, $y > 0$ on (P_0,P_M).

Suppose that at $t = 0$ the population has initial-value

$$P(0) = P_0 > 0 \quad , \tag{2}$$

and that subsequently individuals are harvested at a rate

$$E(t)P(t)$$

with E a nonnegative function representing the (harvesting) effort. Then

$$\dot{P} = y(P) - EP \quad , \tag{3}$$

and using this equation and (2) the yield

$$\int_0^T E(t)P(t)dt$$

on a time interval $[0,T)$ can be written as a functional of P:

$$Y_T(P) = P_0 - P(T^-) + \int_0^T y(P(t))dt \quad . \tag{4}$$

Let us agree to use the term path for a *strictly-positive, piecewise continuous*, right-continuous,[1] piecewise C^1 function P on $[0,\infty)$. Assume that we are given a family A of paths. Then an optimal path relative to A is a path $P^* \in A$ with the following property: given any $P \in A$, $P \neq P^*$,

$$Y_T(P^*) > Y_T(P) \tag{5}$$

for all sufficiently large T (that is, for all T larger than some $T_1 = T_1(P)$). Note that, since (5) is strict, there is at most one optimal path.

There are many possible objective (profit) functions with which one can work.[2] For convenience, we have chose to optimize the total yield. Our analysis goes through, almost without change, when a cost term proportional to

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[1] I.e., $P(t) = P(t^+)$. This assumption removes ambiguity at points of discontinuity.

[2] A thorough discussion of objective functions appropriate to harvesting problems is given by Clark [2].

$$\int_0^T E(t)dt$$

is subtracted from (4).

Remark. The infinte-horizon problem is usually formulated with the aid of a discount factor $e^{-\kappa t}$ ($\kappa > 0$); one then maximizes the total yield

$$\int_0^\infty E(t)P(t)e^{-\kappa t}dt \quad .$$

This formulation renders the yield in the immediate future more important than that of the distant future. To the contrary, our procedure models situations in which one is willing to accept a sub-optimal yield initially, in order to eventually produce the best yield possible. The use of a discount factor may be sound from an economic point of view, but our policy is certainly more appropriate to a long range conservation program. It is interesting to note that discounting generally leads to a smaller size for the ultimate population.

b. Optimal harvesting when the effort is allowed to be unbounded.

Our first step will be to define the class A of paths. To begin with, we require that the effort be nonnegative; in view of (3), this is insured by the constraint:

(A_1) for any $t > 0$ at which P is differentiable,

$$\dot{P}(t) \le y(P(t)) \quad .$$

Next, note that $Y_T(P)$ is well defined for all paths P, even those that suffer jump discontinuities. If individuals are harvested, rather than stocked, the population after such a jump must be less than the population before. We therefore add the second constraint:

(A_2) $P(0) \le P_0$, and for any $t > 0$ at which P jumps,

$$P(t) \le P(t^-) \quad .$$

To interpret the differential equation (3) on paths with jump discontinuities, we rewrite it in the form

$$(\ln P)^{\bullet} = \frac{1}{P} y(P) - E \quad .$$

For this equation to make sense for functions with jumps, E must be interpreted as a distribution. Indeed, if t_k $(k = 1,2,\ldots,n)$ denote the times of discontinuity of P, then

$$E(t) = E_0(t) + \delta(t) \ln \frac{P_0}{P(0)} + \sum_k \delta(t-t_k) \ln \frac{P(t_k^-)}{P(t_k)}$$

with E_0 piecewise continuous and δ the delta distribution. Note that, with E defined in this manner, (A_2) follows from the requirement $E \geq 0$.

Since y has a strict maximum at P_M, we would expect the optimal policy to involve reaching P_M as quickly as possible and then remaining there for all subsequent time. For $P_0 > P_M$ this is easily accomplished by jumping immediately to P_M, since such a jump is consistent with the constraint (A_2). For $P_0 < P_M$ a jump from P_0 to P_M is not possible; as we shall see, here the optimal way to reach P_M is to refrain from harvesting. With this in mind we introduce the following definition: the <u>free-growth curve</u> starting at $Z_0 \in (0,P_M)$ at time $t_0 \geq 0$ is the solution $Z(t)$, $t \geq t_0$, of the initial-value problem

$$\dot{Z} = y(Z) \quad ,$$

$$Z(t_0) = Z_0 \quad .$$

Note that, since $y > 0$ on $(0,P_M]$, Z is strictly-increasing and reaches P_M in finite time.

<u>Theorem.</u>[1] <u>Relative to</u>

$$\mathcal{A} = \{\underline{\text{paths consistent with}}\ (A_1)\ \underline{\text{and}}\ (A_2)\}$$

<u>the optimal path is given by</u>

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[1] Cf., e.g., Clark and Munro [1], Clark [2], Chapter 2.

$$P^*(t) = P_M, \quad t \geq 0 \tag{6}$$

for $P_0 \geq P_M$, and by

$$P^*(t) = \begin{cases} Z(t), & 0 \leq t < t_M \\ P_M, & t_M \leq t < \infty \end{cases} \tag{7}$$

for $P_0 < P_M$. Here Z is the free-growth curve starting at P_0 at time $t = 0$, while t_M is the time at which Z reaches P_M.

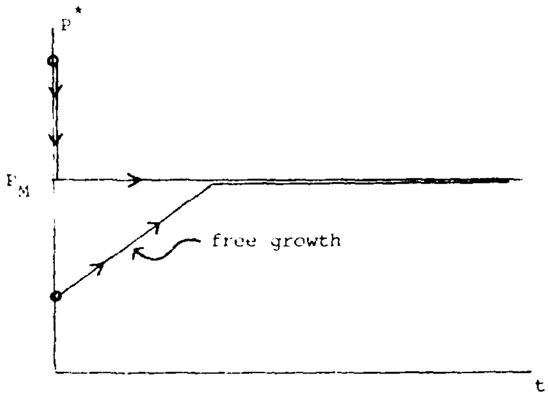

Figure 2 Optimal policies

By (3), the effort E required to hold $P(t)$ constant at P_M is given by

$$E = y(P_M)/P_M.$$

Thus for the path (6),

$$E(t) = y(P_M)/P_M + \delta(t)\ln\frac{P_0}{P_M},$$

while for (7),

$$E(t) = \begin{cases} 0, & 0 \leq t < t_M \\ y(P_M)/P_M, & t_M \leq t < \infty. \end{cases}$$

Our proof of the theorem begins with

Assertion 1. Let $P \in A$ be optimal. Then:

(i) $P(t) \leq P_M$ for all $t \geq 0$;

(ii) $P(t) = P_M$ for all sufficiently large t.

Proof. Assume that (i) is not true, so that $P > P_M$ on some set $J \subset [0,\infty)$. Define $G \in A$ by $G(t) = \min\{P_M, P(t)\}$. Then, since $G \leq P$, it follows from (1) that

$$Y_T(G) - Y_T(P) = \int_{[0,T]\cap J} [y(P_M) - y(P(t))]\,dt + P(T^-) - G(T^-) > 0$$

whenever $T > \inf J$, which contradicts our hypothesis that P be optimal. This proves (i).

Before beginning the proof of (ii) note that, by (1) there exists an $L \in (0, P_M)$ such that

$$P_1 < P_2,\ P_2 \in [L, P_M] \Rightarrow y(P_1) < y(P_2) \quad . \tag{8}$$

Our first step in establishing (ii) will be to show that

$$\lim_{t \to \infty} P(t) = P_M \quad . \tag{9}$$

Assume, to the contrary, that (9) is not true. Then, by (i),

$$\liminf_{t \to \infty} P(t) < P_M \quad ,$$

and there exists a $K \in (L, P_M)$ and an increasing sequence $\{t_n\}$ with $t_n \to \infty$ and

$$P(t_n) < K$$

for all n. Let Z denote the free-growth curve starting at K at $t = 0$, let κ denote the time at which Z reaches P_M, and let

$$Z_n(t) = Z(t - t_n) \quad ,$$

so that Z_n is the free-growth curve starting at K at time t_n.

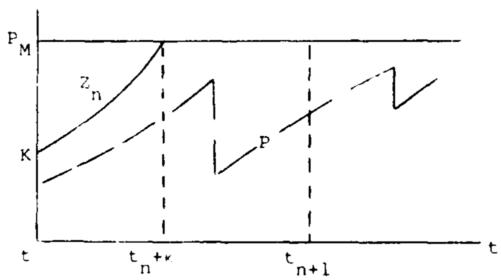

Figure 3. Bounding P by Z_n.

Further assume, without loss in generality that $t_{n+1} > t_n + \kappa$ for all n. (If $\{t_n\}$ does not have this property, some subsequence does.) Since

$$P(t_n) < Z_n(t_n) \quad ,$$

$$\dot{Z}_n = y(Z_n),\ \dot{P} \le y(P) \quad ,$$

and since the jumps in P are downward, we may conclude from a well known comparison theorem[1] for differential inequalities that

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[1] Cf., e.g., Hartman [7], Thm. 4.1, p. 26.

$$P \le Z_n \quad \text{on} \quad [t_n, t_n+\kappa] \quad . \tag{10}$$

Thus, for P^* defined by (6) or (7) and t_n sufficiently large,

$$\int_{t_n}^{t_{n+1}} [y(P^*(t)) - y(P(t))]dt = \int_{t_n}^{t_{n+1}} [y(P_M) - y(P(t))]dt$$

$$\ge \int_{t_n}^{t_n+\kappa} [y(P_M) - y(Z_n(t))]dt = \int_0^{\kappa} [y(P_M) - y(Z(t))]dt = D > 0 \quad ,$$

where we have used (8); hence

$$F(T) = \int_0^T [y(P^*(t)) - y(P(t))]dt \to +\infty \quad \text{as} \quad T \to \infty \quad .$$

Thus, since $P \ge 0$,

$$Y_T(P^*) - Y_T(P) = F(T) + P(T^-) - P_M \to +\infty \quad \text{as} \quad T \to \infty \quad ,$$

which again contradicts the assumed optimality of P. Thus P satisfies (9).

Next, by (8) and (i), there exists a time a for which

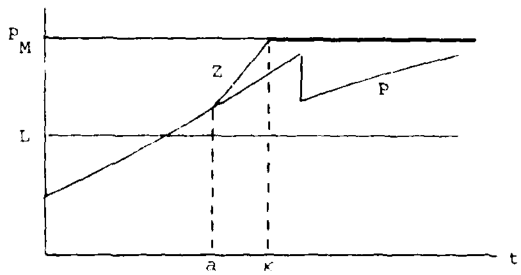

Figure 4. Construction of G.

$P(t) \in [L, P_M]$ for all $t \ge a$. Let Z be the free-growth curve starting at $P(a)$ at time $t = a$, let κ denote the time at which Z reaches P_M, and let $G \in A$ be defined by

$$G(t) = \begin{cases} P(t), & 0 \le t < a \\ Z(t), & a \le t < \kappa \\ P_M, & \kappa \le t < \infty \quad . \end{cases}$$

Then

$$Y_T(G) - Y_T(P) = \int_a^T [y(G(t)) - y(P(t))]dt + P(T^-) - P_M \tag{11}$$

for all $T > \kappa$. Further, the argument leading to (10) here tells us that

$$P \leq Z \quad \text{on} \quad [a,\kappa] \quad .$$

Suppose that (ii) is not satisfied. Then, by (8), (11), and the properties of G,

$$y(G(t)) \geq y(P(t)) \quad \text{for all} \quad t \in [a,\infty) \quad ,$$

the inequality being strict on an open set. Thus, since $P(t) \to P_M$ as $t \to \infty$, (1) must be strictly positive for T sufficiently large. Since this cannot be so, (ii) must be valid.

In view of Assertion 1, it suffices to establish optimality within the class

$$A_M = \{P \in A \mid P \leq P_M \quad \text{on} \quad [0,\infty),\ P(t) = P_M$$
$$\text{for all sufficiently large} \quad t\} \quad .$$

Assume that $P_0 \geq P_M$. Let P^* be given by (6) and choose $P \in A_M$, $P \neq P^*$. Then $P(T) = P^*(T) = P_M$ *for all sufficiently large* T, *and for all such* T,

$$Y_T(P^*) - Y_T(P) = \int_0^T [y(P_M) - y(P(t))]dt > 0 \quad ;$$

hence P^* is optimal.

To complete the proof we must establish the optimality of (7) when

$$P_0 < P_M \quad .$$

With this in mind, we introduce the following notation. Let $[A,C]$ $(0,P_M]$, let Z denote the free-growth curve starting at A at $t = 0$, let κ denote the time at which Z reaches C, and let $[a,c]$ $[0,\infty)$ with

$$c \geq a + \kappa \quad . \tag{12}$$

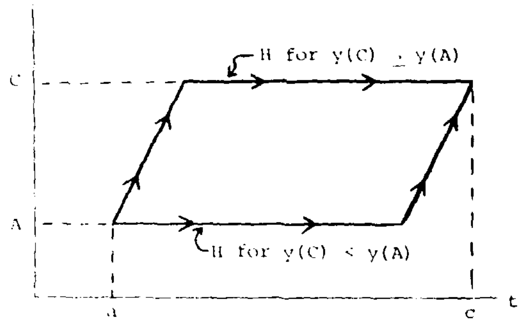

Figure 5. Optimal transitions.

Further, let H be defined on $[a,c]$ by

$$H(t) = \begin{cases} Z(t-a), & a \leq t < a + \kappa \\ C, & a + \kappa \leq t \leq c \end{cases} \tag{13}$$

if $y(C) \geq y(A)$;

$$H(t) = \begin{cases} A, & a \leq t < c - \kappa \\ Z(t-c+\kappa), & c - \kappa \leq t \leq c \end{cases}$$

if $y(C) < y(A)$. We call H the <u>optimal transition</u>[1] from A to C during $[a,c]$; κ is the <u>transition time</u> of H, $c - a - \kappa$ the <u>rest time</u> of H. Finally, a path $P \in A_M$ is a <u>rise</u> during $[a,c]$ if

$$P(a) < P(t) < P(c) \quad \text{for all } t \in (a,c) \quad . \tag{14}$$

<u>Assertion</u> 2. Let $P \in A_M$ be a rise during $[a,c] \subset [0,\infty)$. Then:

(i) There is an optimal transition H between $P(a)$ and $P(c)$ during $[a,c]$.

(ii) If the rest time of H vanishes, then $P = H$ on $[a,c]$.

(iii) If y is monotone on $[P(a),P(c)]$, then

$$y(H(t)) \geq y(P(t)) \quad \text{for all } t \in [a,c] \quad ,$$

with strict inequality on an open set if the rest time of H is nonzero.

<u>Proof</u>. We will give the proof only for

$$y(P(c)) \geq y(P(a)) \quad .$$

Clearly,

$$P(t) \leq Z(t-a), \; a \leq t \leq a + \kappa \tag{15}$$

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[1] Cf., Spence and Starrett [3], p. 393.

(cf. (10)). Thus c is consistent with (12) and the optimal transition (13) is well defined. Assume (for the remainder of this paragraph) that the transition time κ of H equals $c - a$. Then $H(t) = Z(t-a)$ for all $t \in [a,c]$, and since

$$H(c) = P(c) \quad ,$$

$$\dot{H} = y(H),\ \dot{P} \leq y(P) \quad ,$$

it follows[1] that $P \geq H$ on $[a,c]$; this inequality and (15) yield $P = H$ on $[a,c]$.

To prove (iii) let y be monotone and hence strictly increasing on $[P(a),P(c)]$. (The monotonicity is strict since y has at most a finite number of critical points in $(0,P_M)$.) It is clear from (15) and (14) that $P \leq H$ on $[a,c]$, and if the rest time of H is nonzero, that $P(a+\kappa) < H(a+\kappa) = P(c)$. This yields the validity of (iii).

We are now in a position to establish the optimality within A_M and hence A - of the path P^* defined by (7). Thus choose $P \in A_M$, $P \neq P^*$, and let T_M be the least time for which

$$P(t) = P_M \quad \text{for all} \quad t \geq T_M \quad . \tag{16}$$

Since all jumps in P are downward, P must take on all values in $[P_0,P_M]$. Let $C_1,C_2,\ldots,C_n$ with $P_0 = C_1 < C_2 < \cdots < C_n = P_M$

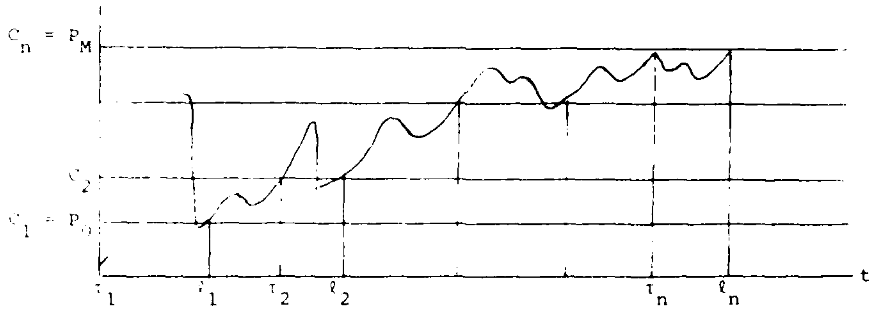

Figure 6. Decomposition of P.

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[1] Cf., e.g., Hartman [7], Remark 1, p. 26.

denote the extreme points of y in $[P_0, P_M]$, let ℓ_k $(k = 1,2,\ldots,n)$ denote the <u>last</u> time in $[0,T_M]$ at which P takes on the value C_k (note that $\ell_n = T_M$), let τ_k $(k = 2,3,\ldots,n)$ denote the <u>first</u> time after ℓ_{k-1} at which P has the value C_k, and define $\tau_1 = 0$. Then

$$P \text{ is a rise during } [\ell_k, \tau_{k+1}] \quad (k = 1,2,\ldots,n-1) \quad . \tag{17}$$

Consider the path $R \in A_M$ obtained from P by replacing each rise (17) by the corresponding optimal transition. Then, since y is monotone on each $[P(\ell_k), P(\tau_{k+1})]$, we may conclude from Assertion 2 that

$$y(R(t)) \geq y(P(t)) \quad \text{for all} \quad t \geq 0 \quad .$$

(Note that P and R coincide on each $[\tau_k, \ell_k]$.) Thus

$$Y_T(R) \geq Y_T(P) \quad \text{for all} \quad T \geq T_M \quad . \tag{18}$$

We shall complete the proof by showing that

$$Y_T(P^*) > Y_T(P) \quad \text{for all } T \geq T_M \quad . \tag{19}$$

For convenience, we consider separately the two cases: $R = P^*$, $R \neq P^*$.

<u>Case</u> 1 $(R = P^*)$. Here

$$\tau_k = \ell_k \ (k = 1,2,\ldots,n) \text{ and the rest time -} \atop \text{for each rise except the last - vanishes} \quad . \tag{20}$$

Thus, by (ii) of Assertion 2, R and P coincide outside of the last rise-interval $[\ell_{n-1}, \tau_n]$. The rest time associated with this last rise cannot vanish, because if it did, then R and P would coincide everywhere, an impossibility, since $R = P^* \neq P$. Consequently, we may conclude from (iii) of Assertion 2 and (8) that

$$Y_T(P) < Y_T(R) \ (= Y_T(P^*)) \quad \text{for all} \quad T \geq T_M \quad .$$

<u>Case</u> 2 $(R \neq P^*)$. Let $T \geq T_M = \ell_n$. By (7), the integral

$$\int_0^T y(P^*(t))dt$$

is equal to $(T-T_M)y(P_M)$ plus the integral of $y(P^*)$ over the free-growth segment of P^*. Since the free-growth equation is autonomous, this latter integral equals the integral of $y(R)$ over the free-growth portions of R. Thus

$$Y_T(P^*) - Y_T(R) = (T-T_M)y(P_M) - I_T(R) \quad , \tag{21}$$

where $I_T(R)$ is the integral of $y(R)$ over the set Ω consisting of the rest portions of the optimal transitions that comprise R and the union of the intervals (τ_1,ℓ_1), $(\tau_2,\ell_2),\ldots,(\tau_n,T)$. On Ω, $y(R) < y(P_M)$ and (as we shall show at the end of the proof) Ω contains an open set $\mathcal{O}$ with

$$y(R(t)) < y(P_M) \quad \text{for all} \quad t \in \mathcal{O} \quad . \tag{22}$$

Thus, since the measure of Ω is $T - T_M$,

$$I_T(R) < (T-T_M)y(P_M) \quad ,$$

and (21) yields $Y_T(P^*) > Y_T(R)$. In view of (18), this inequality implies (19).

To verify (22) note first that (20) cannot be satisfied, for if it were, then R would equal P^*. Thus at least one interval $[\tau_k,\ell_k]$ or at least one rest interval with $R \neq P_M$ has nonzero duration. This interval clearly contains an open set $\mathcal{O}$ on which (22) is satisfied. (If the interval in question is $[\tau_n,\ell_n]$, the existence of $\mathcal{O}$ is insured by the equality of R and P on $[\tau_n,\ell_n]$ and the fact that $T_M = \ell_n$ is the smallest time consistent with (16)).

Remark 1. The construction R - obtained from P by replacing each rise (between extrema of y) by its corresponding optimal transition - can be defined (in an obvious manner) for any function P with domain an interval $[a,b]$, as long as $P(a) < P(b)$ and $P(t) \leq P(b)$ for all $t \in (a,b)$. We will refer to this construction as the *quasi-optimizer of P on $[a,b]$*.

c. Bounded effort.

The problem becomes more complex when the effort is restricted by an upper bound $\bar{E}$. We require that $E(t) \leq \bar{E}$ for all t, or equivalently,

$$(A_3) \qquad \dot{P}(t) \geq y(P(t)) - \bar{E}P(t) \quad .$$

A path which obeys (A_1) and (A_3) is necessarily continuous. We therefore dismiss (A_2) and restrict our attention to paths consistent with (A_1), (A_3), and the initial condition (2).

In addition to assuming that y is a C^1 function on $[0,\infty)$ with a finite number of critical points, we assume

$$\limsup_{P \to \infty} y(P) \leq 0 \quad . \tag{23}$$

Then there is a largest value $P_S (< \infty)$ for which

$$y(P_S) = \bar{E}P_S \quad .$$

Further, by (23), y restricted to $[P_S,\infty)$ has a maximum; we assume that this maximum occurs at exactly one point $P_M \in [P_S,\infty)$:

$$y(P_M) > y(p) \quad \text{for all} \quad p \geq P_S,\ p \neq P_M \quad . \tag{24}$$

Finally, we assume[1] that $y > 0$ on $(0,P_S)$. We use the notation

$$S = y(P_S),\ M = y(P_M) \quad ; \tag{25}$$

then $M > S$.

In addition to free growth curves, the optimal path involves <u>maximal-effort curves</u>; that is, solutions $X(t)$, $0 \leq t < \infty$, of the equation

$$\dot{X} = y(X) - \bar{E}X \quad . \tag{26}$$

We again appeal to a comparison theorem[2] to note that if P satisfies (A3) and $P(t_0) = X(t_0)$, then

$$P \leq X \quad \text{on} \quad [0,t_0],\ P \geq X \quad \text{on} \quad [t_0,\infty) \quad . \tag{27}$$

For $P_0 < P_M$, we will use the following notation:

$$U_1 = \inf\{p > P_0 \mid y(p) > M\} \quad ,$$

and for each value of $n (n = 1,2,\ldots)$ for which the underlying set is nonempty,

$$V_n = \inf\{p > U_n \mid y(p) < M\} \quad ,$$

$$U_{n+1} = \inf\{p > V_n \mid y(p) > M\} \quad .$$

Then $U_n < V_n < U_{n+1}$,

---

[1] This assumption may be weakened as indicated in Footnote 2, p. 2.

[2] Cf., e.g., Hartman [7], Thm. 4.1 and Remark 1, p. 26.

$$y > M \quad \text{on} \quad (U_n, V_n) \quad , \tag{28}$$

and, by (24), $V_n \leq P_S$, for all n. Since y has a finite number of extrema, it is clear that there is a finite number N (say) of U_n's and the same number of V_n's.

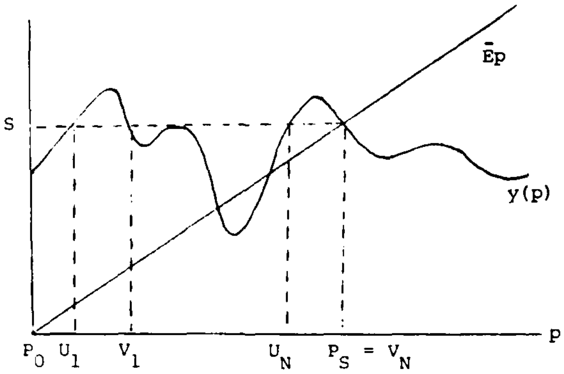

Figure 7. The function y for Case A.

We consider two cases.

Case A. P_M is not a local maximum of y. Then, by (24),

$$P_M = P_S, \; y < S \quad \text{on} \quad (P_S, \infty) \quad , \tag{29}$$

and $V_N = P_S$.

Theorem A. Relative to

$$A = \{\text{paths consistent with } (A_1), (A_3), \text{ and } (2)\}$$

the optimal path is P^*, as described below.

(a) If $P_0 < P_S$, then

$$P^*(t) = \begin{cases} Z_1(t), & 0 < t < u_1 \\ X_1(t), & u_1 < t < v_1 \\ \vdots & \\ Z_N(t), & v_{N-1} < t < u_N \\ X_N(t), & u_N < t < \infty \end{cases} \tag{30}$$

with

Z_1 the free-growth curve through P_0 at $t = 0$,

Z_n the free-growth curve through V_{n-1} at $t = v_{n-1}$,

u_n the time at which $Z_n(u_n) = U_n$,

X_n the maximal-effort curve through U_n at $t = u_n$,

v_n the time at which $X_n(v_n) = V_n$.

(b) If $P_0 > P_S$, then P^* is the maximal-effort curve through P_0 at $t = 0$.

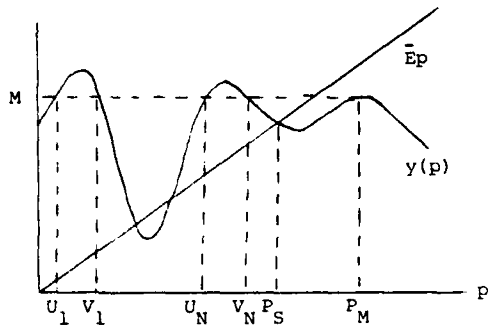

Figure 8. The function y for Case B.

We justify the v_n by noting that, for $n < N$, $y(p) - \bar{E}p > 0$ on $[U_n, V_n]$, so that X_n reaches V_n in a finite time. The optimal paths are graphed in Figure 9. Before proving the theorem for Case A, we state the results for

Case B. P_M is a local maximum for y (Figure 8).

Theorem B. Relative to

$$A = \{\text{paths consistent with } (A_1), (A_3), \text{ and } (2)\}$$

the optimal path is P^*, as described below

(a) If $P_0 < P_M$,

$$P^*(t) = \begin{cases} Z_1(t), & 0 \le t < u_1 \\ X_1(t), & u_1 \le t < v_1 \\ \vdots & \\ Z_{N+1}(t), & v_N \le t < t_M \\ P_M, & t_M \le t < \infty \end{cases}.$$

Here Z_n, X_n, u_n and v_n are interpreted as in Theorem A, Z_{N+1} is the free-growth curve through V_N at $t = v_N$, and t_M is the time at which $Z_{N+1}(t_M) = P_M$.

(b) If $P_0 > P_M > P_S$,

$$P^*(t) = \begin{cases} X(t), & 0 \le t < t_M \\ P_M, & t_M \le t < \infty \end{cases},$$

where X is the maximal-effort curve through P_0 at $t = 0$, and t_M is the time at which $X(t_M) = P_M$. (Since $X(\infty) = P_S < P_M$, $t_M < \infty$.)

(c) If $P_0 > P_M$ and $P_S = P_M$, then P^* is the maximal-effort curve through P_0 at $t = 0$.

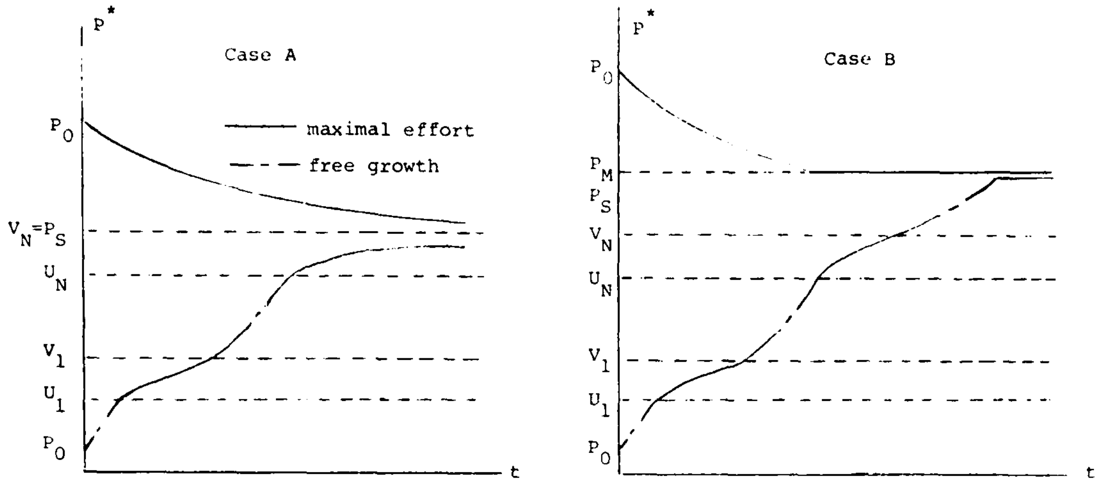

Figure 9. Optimal paths.

We will prove the theorem only for Case A(a). Since the technical aspects of the proof are rather complicated, we offer, as a preliminary, the following intuitive argument in support of the optimality of the path (30).

We first ask, is there an optimal equilibrium value for the population? By (A_3), any equilibrium value p must satisfy $v(p) \leq \bar{E}p$. It is clear from Figure 8 that the "best" such value is $p = P_S$. We therefore expect, and will indeed show, that the optimal path asymptotically approaches P_S.

We next ask, what is the best way to make the population grow from its current size to the optimal size P_S? We know that the optimal asymptotic yield rate is S. Thus if the current yield rate is less than S, we refrain from harvesting to hasten the growth of the population to a size which will produce a higher yield. On the other hand, when the current yield rate is higher than S, we are in an unusually profitable situation. Although the population will inevitably grow beyond this profitable stage, we can, by harvesting with maximal effort, prolong the period of high profit. The path P^* as defined in (30) behaves in the manner just described.

Remark 2. If y has a strict global maximum at $P_M > P_S$, which will be the case if $\bar{E}$ is sufficiently large, then it is possible to harvest so that P is held constant at P_M. For this case Theorem B is completely analogous to our theorem for unbounded effort. For $P_0 < P_M$ the optimal path is a free-growth curve followed by $P(t) \equiv P_M$; for $P_0 > P_M$ the optimal path is a maximal-effort curve (which replaces the jump to P_M) followed by $P(t) \equiv P_M$.

We now begin our proof of Theorem A(a). Thus let

$$P_0 < P_S \quad .$$

Assertion 3. Let $P \in A$ be optimal. Then:

(i) $\lim_{t\to\infty} P(t) = P_S$;

(ii) P is nondecreasing.

Proof. Note that $P(t) \le P_S$, $0 \le t < \infty$, since otherwise, the path $Q \in A$ defined by

$$Q(t) = \min\{P(t), P_S\}$$

would be better. (For convenience, we use the terminology: Q is better than P if $Y_T(Q) > Y_T(P)$ for all sufficiently large T.) Assume first that $P(t_0) \in [U_n, V_N]$ at some time t_0. Let X be the maximal-effort curve through $P(t_0)$ at t_0. By (26), $X(t) \to P_S$ as $t \to \infty$; thus, since $X \le P \le P_S$ on $[t_0, \infty)$, (i) is satisfied. Assume next that $P(t) < U_N$ for all t. Let

$$S_e = \{p \in [0, U_N] \mid y(p) \ge s - e\} \quad . \tag{31}$$

Then there exists an $e > 0$ such that

$$y(p) > \bar{E}p \quad \text{for all} \quad p \in S_e \quad . \tag{32}$$

Moreover, since y has at most a finite number of critical points, S_e is the union of a finite number of compact intervals. By (26), (27), and (32), if P enters any such interval, it remains only a finite time, and it never returns. Thus $P(t)$ lies outside S_e for all sufficiently large t, and $(25)_1$ and (31) yield

$$y(P_S) - y(P(t)) > e \tag{33}$$

for all such t. Let

$$Q(t) = \begin{cases} Z(t), & 0 \leq t \leq t_S \\ P_S, & t_S \leq t < \infty \end{cases},$$

where Z is the free-growth curve starting at P_0 at $t = 0$, while t_S is the time at which Z reaches P_S. Then a trivial computation, based on (33), shows that

$$\lim_{T\to\infty} [Y_T(Q) - Y_T(P)] = +\infty \quad ,$$

so that Q is better than P. *This contradicts the optimality of P and the proof of (i) is complete.*

(ii) We will show that $\dot{P}(t^+) \geq 0$ for all t. Assume, to the contrary, that $\dot{P}(t_1^+) < 0$ at some time t_1, and let $P_1 = P(t_1)$. Then by (A_3), $y(P_1) < \bar{E}P_1$, and, since $y(P_S) = \bar{E}P_S$, $P_1 < P_S$. In addition,

$$q = \sup\{p \in [0,P_1) \mid y(p) = \bar{E}p\} \quad ,$$

$$\tau = \inf\{p \in (P_1,P_S] \mid y(p) = \bar{E}p\}$$

exist (recall $y(0) \geq 0$), and

$$\begin{aligned} &y(p) < \bar{E}p \quad \text{for all} \quad p \in (q,\tau) \quad , \\ &0 \leq q < P_1 < \tau < P_S \quad . \end{aligned} \tag{34}$$

(Cf. Figure 10. The inequality $\tau < P_S$ is a consequence of (28) and $(34)_1$).

Let X be the maximal-effort curve starting at P_1 at $t = t_1$. Since the maximal-effort curve through q remains constant at q, $X > q$ on $[t_1,\infty)$; hence (27) yields the conclusion

$$P > q \quad \text{on} \quad [t_1,\infty) \quad . \tag{35}$$

Let $t_1 + \ell$ be the first time after t_1 at which $P(t_1+\ell) = P_1$. (Since $P_1 < P_S$, the existence of $\ell > 0$ is assured by (i).) Then

$$P < P_1 \quad \text{on} \quad (t_1,t_1+\ell) \quad . \tag{36}$$

By $(34)_2$, (35), and (36), $q < P(t) < \tau$ on $(t_1,t_1+\ell)$, so that, by $(34)_1$,

$$y(P(t)) < \bar{E}P(t) < \bar{E}\tau = y(\tau) \quad \text{for all} \quad t \in (t_1,t_1+\ell) \quad .$$

Thus the path $Q \in A$, pictured in Figure 11 and defined by

$$Q(t) = \begin{cases} Q(t), & 0 \le t < t_1 \\ P(t+\ell), & t_1 \le t < t_2 - \ell \\ \tau, & t_2 - \ell \le t < t_2 \\ P(t), & t_2 \le t < \infty \end{cases},$$

where t_2 ($> t_1+\ell$) is the first time after t_1 at which $P(t_2) = \tau$, is better than P. Hence $\dot{P}(t_1^+) < 0$ is not possible. This completes the proof of Assertion 3.

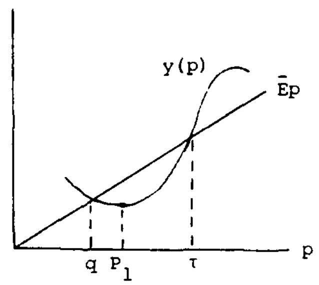

Figure 10

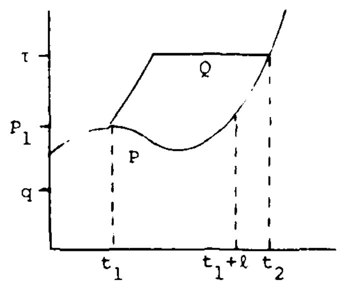

Figure 11

The following definition will be useful. Let f be a continuous, nondecreasing function on $[0,\infty)$, and let F belong to the range of f. Then f <u>stretched at</u> F <u>by amount</u> $\ell > 0$ is the function $\vec{f}$ defined by

$$\vec{f}(t) = \begin{cases} f(t), & 0 \le t < t_0 \\ F, & t_0 \le t < t_0 + \ell \\ f(t-\ell), & t_0 + \ell \le t < \infty \end{cases},$$

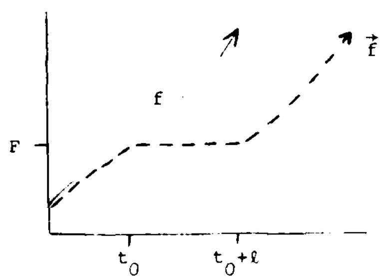

Figure 12. f stretched at F.

where t_0 is the first time at which $f(t_0) = F$. A typical stretched function is shown in Figure 12. Of course, by successive applications of this definition we may stretch a function at a finite number of points in its range.

We are now in a position to establish the optimality of the path P^* defined by (30). In view of Assertion 3, it suffices to show that P^* is better than any path $P \in \mathcal{A}$, $P \neq P^*$, consistent with (i) and (ii) of Assertion 3. Let P be such a path, let $\hat{u}_n (n = 1,2,\ldots,N)$ denote the time at which P crosses U_n, and let $\hat{v}_n(n = 1,2,\ldots,N-1)$ denote the time at which P crosses V_n. (These times are unique; indeed, if t denotes any one of these times, then $\dot{P}(t) > y(P(t)) - \overline{E}P(t) > 0$.) Let $\vec{P}$ be P stretched at $V_1, V_2, \ldots, V_{N-1}$ with $\lambda_n - (\hat{v}_n - \hat{u}_n)$ the amont of the stretch at V_n. Here λ_n is the time it takes the maximal-effort curve starting at U_n to reach V_n. Finally, let $\tilde{u}_n$ $(n = 1,2,\ldots,N)$ be the time at which $\vec{P}$ crosses U_n, so that $\tilde{v}_n = \tilde{u}_n + \lambda_n$ $(n = 1,2,\ldots,N-1)$ is the last time $\vec{P}$ takes on the value V_n. Then, since $P \neq P^*$,

$$\vec{P} \neq P^* \quad ; \tag{37}$$

in addition, a simple analysis shows that

$$\lim_{T\to\infty} [Y_T(\vec{P}) - Y_T(P)] = 0 \quad .$$

It therefore suffices to show that

$$P^* \text{ is better than } \vec{P} \quad . \tag{38}$$

Consider the path R obtained from $\vec{P}$ as follows:

(I) On $[\tilde{u}_n, \tilde{v}_n]$ replace $\vec{P}$ by the maximal-effort curve starting at U_n at $t = \tilde{u}_n$. (Note that $\vec{P}$ is defined so that this maximal-effort curve reaches V_n at $\tilde{v}_n$.)

(II) On $[\tilde{v}_{n-1}, \tilde{u}_n]$ replace $\vec{P}$ by its quasi-optimizer (cf. Remark 1). Here (I) and (II) hold for $n = 1,2,\ldots,N$ with $v_0 = 0$ and $\tilde{v}_N = \infty$.) The paths $\vec{P}$ and R are shown in Figure 13. Note that these paths generally do not belong to $\mathcal{A}$.

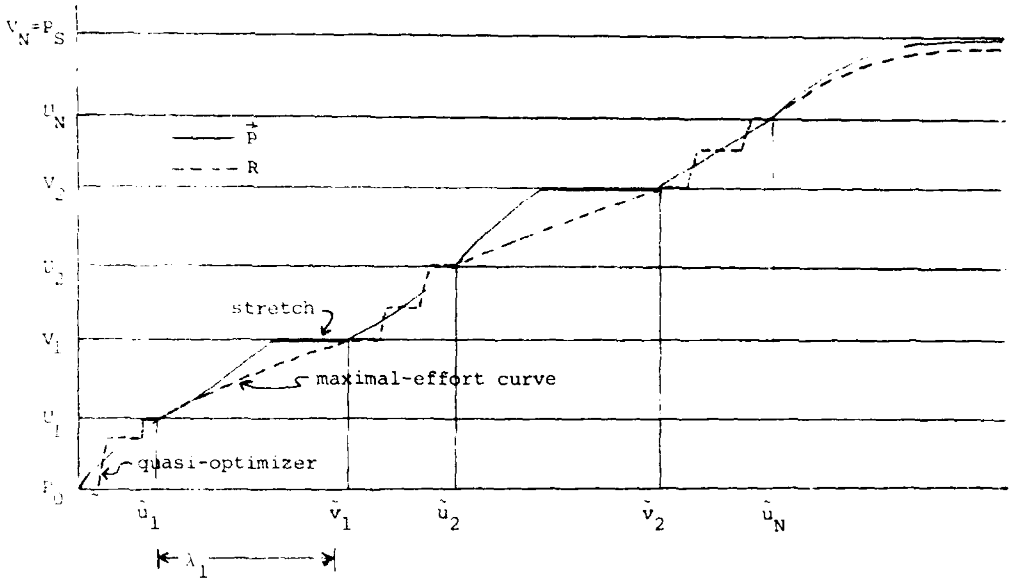

Figure 13. The curves $\vec{P}$ and R.

Assertion 4.

$$\liminf_{T \to \infty} [Y_T(R) - Y_T(\vec{P})] \geq 0$$

with strict inequality if $R \neq \vec{P}$.

Proof. Let $T > \tilde{u}_N$ and define

$$I = \bigcup_{n=1}^{N} [\tilde{v}_{n-1}, \tilde{u}_n], \quad J = \bigcup_{n=1}^{N} [\tilde{u}_n, \tilde{v}_n], \quad J_T = J \cap [0,T] \quad ,$$

$$\mathcal{I}(\Lambda) = \int_\Lambda [y(R(t)) - y(\vec{P}(t))]dt \quad , \tag{39}$$

so that $[0,T] = I \cup J_T$ and

$$Y_T(R) - Y_T(\vec{P}) = \vec{P}(T) - R(T) + \mathcal{I}(I) + \mathcal{I}(J_T) \quad .$$

Since $\vec{P}(\tilde{u}_N) = R(\tilde{u}_N)$, $(27)_2$ implies $\vec{P}(T) > R(T)$. Thus to prove Assertion 4 it suffices

to show that (40) and (41) below are valid:

$$I(I) \geq 0 \text{ with strict inequality if } R \neq \vec{P} \text{ on } I \;; \tag{40}$$

$$\liminf_{T \to \infty} I(J_T) \geq 0 \text{ with strict inequality if } R \neq \vec{P} \text{ on } J \;. \tag{41}$$

Conclusion (40) is an immediate consequence of (ii) and (iii) of Assertion 2 and the fact that $R \neq \vec{P}$ on I only if at least one of the optimal transitions associated with R has a nonzero rest time.

We now prove (41). On an interval $[\tilde{u}_n, \tilde{v}_n]$, R is a maximal effort curve. Because $\dot{\vec{P}} > y(\vec{P}) - \overline{E\vec{P}} > 0$ whenever $U_n < \vec{P} < V_n$, $\vec{P}$ must be strictly increasing on $[\tilde{u}_n, \tilde{v}_n]$ until the time at which $\vec{P}$ reaches V_n. Of course, if $n = N$, $\vec{P}$ may never reach V_N, in which case $\vec{P}$ is strictly increasing on $[u_N, \infty)$. Let C_k, $U_n = C_1 < C_2 < \ldots < C_K = V_n$, denote the extrema of y on $[U_n, V_n]$, so that K is odd, and

$$\begin{aligned} &\text{for } k \text{ even, } y \text{ is strictly increasing on } [C_{k-1}, C_k] \;, \\ &\qquad\qquad \text{strictly decreasing on } [C_k, C_{k+1}] \;. \end{aligned} \tag{42}$$

Let t_k be the time at which $R(t_k) = C_k$. Further, let Q be the function obtained by stretching $\vec{P}$ at each C_k for odd $k < K$, and then restricting the result to $[\tilde{u}_n, \tilde{v}_n]$. The amounts of the stretches are uniquely determined by the requirement $Q(t_k) = R(t_k)(= C_k)$ for k even. It is clear from this construction and (27) that, for k even,

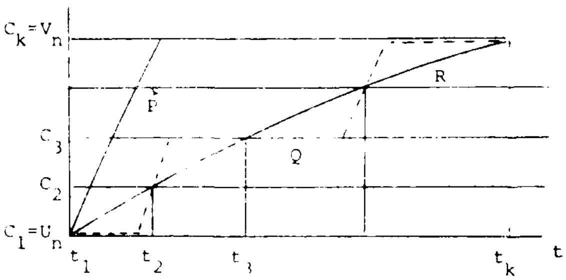

Figure 14. The function Q.

$$C_{k-1} \leq Q \leq R \leq C_k \text{ on } [t_{k-1}, t_k], \; C_k \leq R \leq Q \leq C_{k+1} \text{ on } [t_k, t_{k+1}] \;.$$

These inequalities and (42) imply that

$$y(R(t)) \geq y(Q(t)) \text{ on } [\tilde{u}_n, \tilde{v}_n] \;. \tag{43}$$

The integral of $Y(\vec{P})$ over the times at which $\vec{P}$ is strictly increasing is equal to the analogous integral for $y(Q)$. Thus, since $y(C_k) > y(C_K)$,

$$\int_{\tilde{u}_n}^{\tilde{v}_n} [y(Q(t)) - y(\vec{P}(t))]dt > 0 \quad ,$$

and we conclude from (43) that

$$\int_{\tilde{u}_n}^{\tilde{v}_n} [y(R(t)) - y(\vec{P}(t))]dt > 0 \quad , \tag{44}$$

for $n < N$. Thus to show that $\liminf I(J_T) > 0$ we have only to show that

$$\liminf_{T \to \infty} \int_{\tilde{u}_n}^{T} [y(Q(t)) - y(\vec{P}(t))]dt > 0 \quad . \tag{45}$$

To verify (45) let ℓ_k (k odd) be the length of the stretch in Q at C_k and let L denote the sum of the ℓ_k. Then the integral in (45) is equal to

$$\int_{T-L}^{T} [S - y(\vec{P}(t))]dt + \sum_{\text{odd } k} \ell_k [y(C_k) - S] \quad .$$

The integral tends to zero as $T \to \infty$, since $\vec{P}(t) \to P_S$ as $t \to \infty$, while the sum is > 0, since $y(C_k) > S$. Thus (45) holds.

The remainder of the proof of (41), that $\liminf I(J_T) > 0$ when $R \neq \vec{P}$ on J, can safely be omitted. This completes our proof of Assertion 4.

We now return to the proof of (38) and consider separately two cases.

Case 1 $(R = P^*)$. Here, by (37), $R \neq \vec{P}$, and (38) follows from Assertion 4 with $R = P^*$.

Case 2 $(R \neq P^*)$. Choose $T > \tilde{u}_N$ $(> u_N)$ and consider the integral

$$I_0 = \int_0^T (y(P^*(t)) - y(R(t)))dt \quad .$$

The integral of $y(R)$ over the free-growth portions of R is equal to the analogous integral for P^*. Similarly, the integral of $y(R)$ over the maximal-effort segments of R on $[0,\tilde{u}_N]$ equals its counterpart for P on $[0,u_N]$. Thus $I_0 = I_2 - I_1 - I_3$, where

$$I_1 = \int_{\tilde{u}_N}^{T} y(R(t))dt, \quad I_2 = \int_{\tilde{u}_N}^{T} y(P^*(t))dt$$

and I_3 is the integral of $y(R)$ over those subintervals of I (cf. $(39)_1$) on which P is constant. The measure of these sub-intervals is

$$\tau = \tilde{u}_N - u_N \quad .$$

Thus, since $y(R(t)) \leq S$ on I, we have the bound

$$I_3 \leq \tau S - \Lambda$$

with

(1) $\Lambda = 0$ when $y(R(t)) = S$ on all of the above subintervals,

(2) $\Lambda > 0$ when $y(R(t)) < S$ on at least one such subinterval.

Finally, since $R(t+\tau) = P^*(t)$ for $t > u_N$,

$$I_2 - I_1 = \int_{T-\tau}^{T} y(P^*(t))dt \quad .$$

In view of the above remarks,

$$Y_T(P^*) - Y_T(R) \geq P^*(T-\tau) - P^*(T) + \int_{T-\tau}^{T} [y(P^*(t)) - S]dt + \Lambda \quad ,$$

and, since $P^*(t) \to P_S$,

$$\liminf_{T \to \infty} [Y_T(P^*) - Y_T(R)] \geq \Lambda \geq 0 \quad . \tag{46}$$

For $\Delta > 0$ this result and Assertion 4 imply (38). Thus assume $\Delta = 0$. Since $R \neq P^*$ there is at least one interval Ω in I with R constant at S, so $y(R) - \bar{E}R = S - \bar{E}R > 0$ on Ω. Since $\vec{P} = P \in A$ on I and R fails to satisfy (A3) on $\Omega \subset I$, $\vec{P} \neq R$ on Ω. Thus (38) follows from (46) and Assertion 4. This completes the proof.

d. Discounting.

The usual method of treating an infinite horizon problem is to introduce a discount factor $e^{-\kappa t}$ into the yield. The total yield

$$\int_0^\infty e^{-\kappa t} E(t)P(t)dt$$

is then defined for all bounded paths P and may be written as

$$Y(P) = \int_0^\infty e^{-\kappa t} w(P(t))dt + P_0$$

where

$$w(P) = y(P) - \kappa P \quad . \tag{47}$$

The corresponding optimal harvesting problem for unconstrained effort consists in maximizing Y *over* the class of all bounded paths which are consistent with (A1) and (A2). If w is a C^1 function on $[0,\infty)$ with a strict global maximum at $P_M > 0$ and if

$$w \text{ has no other extrema on } (0,\infty) \tag{48}$$

then the solution of this problem is analogous to the solution of the nondiscounted problem of section b. (cf., e.g., Clark and Munro [1] and Clark [2], chapter 2.) We now show, by example, that if the rather restrictive assumption (48) is eliminated, the optimal path need not be a most rapid approach to P_M.

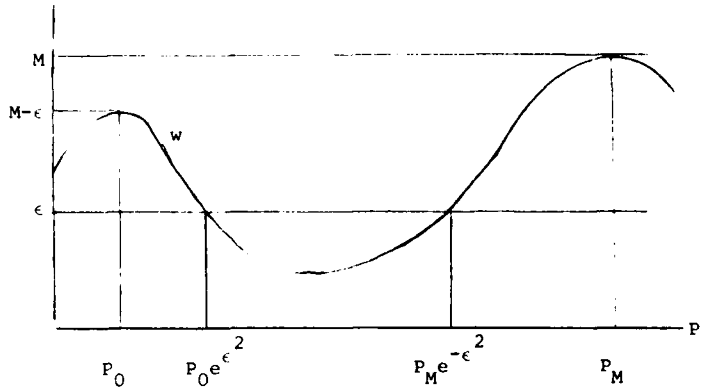

Figure 15. The function w.

Let $P_0 > 0$, $\kappa > 0$, and $M > 0$ be fixed. Further, let $\epsilon > 0$ be arbitrarily small, let $P_M = P_M(\epsilon) = P_0/\epsilon$, and let $w = w_\epsilon$ be a C^1 function on $[0,\infty)$ with the following properties: w has a strict global maximum at P_M, $w > 0$ on $[P_0,P_M]$,

$$w(P_M) = M, \quad w(P_0) = M - \epsilon \quad , \tag{49}$$

and

$$w \le \begin{cases} M - \epsilon & \text{on} \quad [P_0, P_0 e^{\epsilon^2}] \\ \epsilon & \text{on} \quad [P_0 e^{\epsilon^2}, P_M e^{-\epsilon^2}] \end{cases} . \tag{50}$$

we will show that <u>for</u> ϵ <u>sufficiently small the path</u> $\hat{P} \equiv P_0$ <u>is better than the path</u> P^* <u>defined in</u> (7).

To see this let $Z = Z_\epsilon$ be the free-growth curve starting from P_0 at $t = 0$, so that, by (47),

$$\dot{Z} = w(Z) + \kappa Z \quad . \tag{51}$$

Let $t_M = t_M(\epsilon)$ be the time at which Z reaches P_M (cf. (7)). Further, let $t_0 = t_0(\epsilon)$ and $t_1 = t_1(\epsilon)$ denote the times at which Z reaches $P_0 e^{\epsilon^2}$ and $P_M e^{-\epsilon^2}$, respectively, so that $t_0 < t_1 < t_M$. Assume that $\epsilon < \kappa$. Then (51) yields

$$\epsilon Z \le \dot{Z} \le M + \kappa Z \quad \text{on} \quad [0,t_M] \quad . \tag{52}$$

Since M and κ are independent of ϵ, while $P_M e^{-\epsilon^2}$ becomes infinite as $\epsilon \to 0$, we may conclude from the second inequality in (52) that

$$t_1 \to \infty \text{ as } \epsilon \to 0 \quad . \tag{53}$$

Similarly, the first inequality in (52) yields $Z(t_0) > P_0 e^{\epsilon t_0}$, and, since $Z(t_0) = P_0 e^{\epsilon^2}$, it follows that $t_0 < \epsilon$; thus

$$t_0 \to 0 \text{ as } \epsilon \to 0 \quad . \tag{54}$$

In view of (50), $w(Z(t))$ is bounded above by $M - \epsilon$ for $0 < t < t_0$, by ϵ for $t_0 < t < t_1$, and by M for $t_1 < t < t_M$; thus, using (7) and $(49)_2$,

$$Y(P^*) - Y(\hat{P}) < \int_{t_0}^{t_1} (2\epsilon - M)e^{-\kappa t}dt + \int_{t_1}^{\infty} \epsilon e^{-\kappa t}dt = \frac{(2\epsilon - M)e^{-\kappa t_0} + (M - 3\epsilon)e^{-\kappa t_1}}{\kappa} \quad .$$

By (53) and (54) the right side of the above relation has the limit $-M/\kappa$ as $\epsilon \to 0$. Hence we have only to choose ϵ sufficiently small to insure $Y(\hat{P}) > Y(P^*)$.

3. The age-dependent theory without harvesting.

a. Basic equations. Persistent age distributions.

We work within the framework of the nonlinear theory developed in [5]. This theory is based on the equations

$$\begin{aligned}
&\frac{\partial \rho(a,t)}{\partial a} + \frac{\partial \rho(a,t)}{\partial t} + \mu(a,P(t))\rho(a,t) = 0 \quad , \\
&P(t) = \int_0^\infty \rho(a,t)da \quad , \\
&B(t) = \rho(0,t) = \int_0^\infty \beta(a,P(t))\rho(a,t)da \quad ,
\end{aligned} \tag{55}$$

where

$\rho(a,t)$ is the age distribution (the number of individuals, per unit age, of age $a > 0$ at time $t > 0$);

$P(t)$ is the total population;

$B(t)$ is the birth-rate;

μ is the <u>death function</u> ($\mu(a,P)da$ is the probability of dying in the age interval $(a,a+da)$ when the population is P);

β is the <u>birth function</u> ($\beta(a,P)$ is the expected number of births to a person of age a, per unit time, when the population is P).

Here, however, we assume that dependence on total population is confined to the death function. More specifically, we assume that[1]

$$\begin{aligned} \beta(a,P) &= \beta(a) \ , \\ \mu(a,P) &= \mu_n(a) + \mu_e(P) \ . \end{aligned} \tag{56}$$

In $(56)_2$, $\mu_n(a)da$ represents the probability of dying of <u>natural causes</u> during $(a,a+da)$, while $\mu_e(P)da$ is the probability of death due to <u>environmental factors</u> (crowding, etc.) during the same interval. In view of this interpretation,

$$\pi(a) = \exp\{-\int_0^a \mu_n(\alpha)d\alpha\}$$

is the probability of living to age a in a perfect environment; that is, an environment with

$$\mu_e = 0 \ . \tag{57}$$

Assumption $(56)_2$ is special, as the effects of the environment are independent of age; it might be appropriate, for example, to a population in the presence of predators which indiscriminatly eat individuals of all ages.

We assume μ_n, μ_e, and β are C^1 functions on $[0,\infty)$ with $\beta \in L^1(0,\infty)$ and

$$\int_0^\infty \pi(a)\beta(a)da > 1 \ . \tag{58}$$

---

[1] These assumptions with $\mu_n = 0$ and $\beta(a)$ a sum of terms of the form $ba^k e^{-\alpha a}$ were utilized by Gurtin and MacCamy [5,8] to reduce the system (55) to ordinary differential equations. The more general form (56) was introduced by Coleman and Simmes [4].

The left side of (58) represents the net reproduction rate[1] in a perfect environment. A consequence of (58) is that in such an environment the population ultimately grows. This assumption therefore yields a population which is amenable to harvesting.

Because of (58), the equation

$$\int_0^\infty \pi(a)\beta(a)e^{-ra}da = 1$$

has exactly one real solution r, and $r > 0$. We call r the <u>natural growth rate</u>. For a <u>perfect environment</u> the function

$$\rho(a,t) = C\pi(a)e^{-ra}P(t) \tag{59}$$

with

$$P(t) = P(0)e^{rt}$$

and C chosen so that

$$C\int_0^\infty \pi(a)e^{-ra}da = 1 \tag{60}$$

is a solution of (55) ((56) and (57)). We call such solutions <u>persistent age distributions</u>. Their importance is that they represent the asymptotic behavior of general solutions for large time:[2] given any solution ρ there is a constant C_0 such that, for each a,

$$\rho(a,t) \sim C_0\pi(a)e^{-ra}e^{rt}$$

as $t \to \infty$.[3]

---

1 Cf., e.g., Keyfitz [9], p. 102.

2 Feller [10].

3 $f(t) \sim g(t)$ as $t \to \infty$ signifies that $f(t)/g(t) \to 1$ as $t \to \infty$.

b. <u>General solution.</u>[1]

Consider now the initial-value problem consisting of the system (55) supplemented by the initial condition

$$\rho(a,0) = \varphi(a) \tag{61}$$

with initial data φ continuous and of compact support. To solve this problem we write

$$\xi(a,t) = \rho(a,t)\exp\{ \int_0^t \mu_e(P(\lambda))d\lambda\} \quad . \tag{62}$$

Then, because of (56), equations $(55)_{1,3}$ and (61) are equivalent to

$$\begin{gathered} \frac{\partial\xi(a,t)}{\partial a} + \frac{\partial\xi(a,t)}{\partial t} + \mu_n(a)\xi(a,t) = 0 \quad , \\ \xi(0,t) = \int_0^\infty \beta(a)\xi(a,t)da \quad , \\ \xi(a,0) = \varphi(a) \quad . \end{gathered} \tag{63}$$

The equations (63) constitute a linear system for ξ, a system which is, in fact, the classical formulation of the problem for a perfect environment. The corresponding birth-rate

$$B(t) = \xi(0,t)$$

is therefore a solution of the Sharpe-Lotka equation

$$B(t) = \int_0^t \beta(a)\pi(a)B(t-a)da + \Phi(t) \quad , \tag{64}$$

where Φ depends only on the initial data:

$$\Phi(t) = \int_t^\infty \beta(a)\pi(a-t,a)\varphi(a-t)da \quad ,$$

$$\pi(\alpha,a) = \frac{\pi(a)}{\pi(\alpha)} \quad .$$

---

[1] Coleman and Simmes [4].

Once $\mathcal{B}$ is known the total population

$$\mathcal{P}(t) = \int_0^\infty \xi(a,t)da \tag{65}$$

corresponding to ξ is easily calculated using the relations

$$\begin{aligned} \mathcal{P}(t) &= \int_0^t \pi(a)\mathcal{B}(t-a)da + \Psi(t) \quad , \\ \Psi(t) &= \int_t^\infty \pi(a-t,a)\varphi(a-t)da \quad . \end{aligned} \tag{66}$$

It is important to emphasize that ξ, $\mathcal{B}$, and $\mathcal{P}$ are the age distribution, birth-rate, and total population that would prevail in a perfect environment.

To derive (64) and (66) from (63) we integrate $(63)_1$ along characteristics $(t = a +$ constant). The result,

$$\xi(a,t) = \begin{cases} \pi(a)\mathcal{B}(t-a), & 0 \leq a < t \\ \pi(a-t,a)\varphi(a-t) & t \leq a \end{cases} \quad , \tag{67}$$

when combined with $(63)_2$ and (65), yields (64) and (66).

Next, we integrate (62) with respect to age and use $(55)_2$ and (65) to conclude that

$$\mathcal{P}(t) = P(t)\exp\{\int_0^t \mu_e(P(\lambda))d\lambda\} \quad . \tag{68}$$

If we differentiate this relation with respect to t and divide the result by $\mathcal{P}(t)$, we find that P satisfies the ordinary differential equation

$$\dot{P} = [\gamma - \mu_e(P)]P \quad . \tag{69}$$

Here

$$\gamma(t) = \frac{\dot{\mathcal{P}}(t)}{\mathcal{P}(t)} \tag{70}$$

represents the instantaneous growth rate in a perfect environment. Of course, P satisfies the initial condition

$$P(0) = P_0 = \int_0^\infty \varphi(a)da \quad . \tag{71}$$

The above analysis yields the following procedure for solving the initial-value problem (55), (56), and (61):

(a) Solve the linear integral equation (64) for $\mathcal{B}(t)$.

(b) Compute $\mathcal{P}(t)$ from (66) and $\gamma(t)$ from (70).

(c) Solve the differential equation (69) - with initial condition (71) - for $P(t)$.

(d) Compute $\xi(a,t)$ from (67) and $\rho(a,t)$ from (62).

4. <u>Harvesting of age-structured populations</u>.

a. <u>Basic equations</u>. <u>The optimal harvesting problem</u>.

We consider harvesting with <u>effort</u> $E(t)$ independent of age.[1] Thus individuals of age a are harvested at a rate $E(t)\rho(a,t)$, per unit age and time, and, in place of $(55)_1$, we have the balance equation

$$\frac{\partial\rho(a,t)}{\partial a} + \frac{\partial\rho(a,t)}{\partial t} + [\mu(a,P(t)) + E(t)]\rho(a,t) = 0 \quad . \tag{72}$$

The underlying equations are therefore (72) and $(55)_{2,3}$ supplemented by the initial condition (61). (We continue to assume that μ and β satisfy (56).)

We will use the procedure discussed in Section 3 to reformulate the problem. We therefore define

$$\xi(a,t) = \rho(a,t)\exp\{\int_0^t [\mu_e(P(\lambda)) + E(\lambda)]d\lambda\} \quad . \tag{73}$$

Then ξ satisfies the linear system (63), $\mathcal{B}(t) = \xi(0,t)$ is again the solution of the linear integral equation (64), and $\mathcal{P}(t)$, defined by (65), is again given by (66). It is

---

1 Cf. Remark 3.

important to note that ξ, $\mathcal{B}$, and $\mathcal{P}$ represent the age distribution, birth-rate, and total population that would prevail in the absence of harvesting and environmental effects.

As before, the behavior of the actual population $P(t)$ is governed by an ordinary differential equation. Indeed, the exact same steps used to derive (69) now lead to

$$\dot{P} = y(t,P) - EP \tag{74}$$

with

$$y(t,P) = [\gamma(t) - \mu_e(P)]P \quad . \tag{75}$$

In view of (74), the yield

$$\int_0^T \int_0^\infty E(t)\rho(a,t)\,da\,dt = \int_0^T E(t)P(t)\,dt \tag{76}$$

on a time interval $[0,T)$ is given by the functional

$$Y_T(P) = P_0 - P(T^-) + \int_0^T y(t,P(t))\,dt \quad . \tag{77}$$

Roughly speaking, our optimal harvesting problem consists in maximizing (77) subject to certain constraints. This problem can be attacked using the following procedure:

(a) Solve the linear integral equation (64) for $\mathcal{B}(t)$.

(b) Compute $\mathcal{P}(t)$ from (66) and $\gamma(t)$ from (70).

(c) Solve the optimal control problem defined by the differential equation (74) and the functional (76).

Unfortunately, because of the dependence of $y(t,P)$ on t, the optimization problem of (c) does not fit within the framework discussed in Section 1. There is, however, an important class of problems for which this dependence disappears. This class corresponds to initially persistent age distributions, and we shall study it in the next section.

b. Optimal harvesting for initially persistent age distributions.

As noted at the end of Section 3a, when the environment is perfect and harvesting absent, the age distribution ξ is ultimately persistent. Therefore, if the population has been evolving over a long period of time, and if all past harvesting has been with effort

independent of age, then it seems reasonable to assume[1] that initially ρ will have the age structure indicated in (59). We therefore assume that the population is <u>initially persistent</u> in the sense that

$$\varphi(a) = CP_0\pi(a)e^{-ra} \quad ,$$

where C and P_0 are constants with C chosen so that (60) holds, while r is the natural growth rate. The initial-value problem (63) then has the unique solution

$$\begin{aligned} \xi(a,t) &= C\pi(a)e^{-ra}P(t) \quad , \\ P(t) &= P_0e^{rt} \quad , \end{aligned} \tag{78}$$

so that, by (70),

$$\gamma(t) = r \quad . \tag{79}$$

Further, the counterpart of (68) in the present circumstances is simply (68) with $E(\lambda)$ added to $\mu_e(P(\lambda))$; thus $(78)_1$ and (73) imply that

$$\rho(a,t) = C\pi(a)e^{-ra}P(t) \quad . \tag{80}$$

In view of (75) and (79), $y(t,P)$ <u>is independent of</u> t:

$$y(P) = [r - \mu_e(P)]P \quad . \tag{81}$$

Thus the differential equation (74) reduces to

$$\dot{P} = y(P) - EP \quad , \tag{82}$$

while the functional (77) takes the form

$$Y_T(P) = P_0 - P(T^-) + \int_0^T y(P(t))dt \quad . \tag{83}$$

Comparing (82) and (83) with (3) and (4), we see that <u>for an initially persistent population the optimal harvesting problem reduces to the (age-independent) problem discussed in Section</u> 2. Therefore, if y satisfies the hypotheses of Section 2, the theorems of Sections 2b and 2c hold.

---

[1] Cf. (73).

Remark 3. Our results extend trivially to situations in which individuals of age a have an economic value $g(a)$, where g is a nonnegative, L^∞ function on $[0,\infty)$. In this case the yield (76) is replaced by

$$\int_0^T \int_0^\infty g(a)E(t)\rho(a,t)\,da\,dt = \int_0^T E(t)G(t)\,dt \quad , \tag{84}$$

where

$$G(t) = \int_0^\infty g(a)\rho(a,t)\,da \quad ,$$

and (80) implies that

$$G(t) = (C\int_0^\infty g(a)\pi(a)e^{-ra}da)P(t) = C_1P(t)$$

with $C_1 > 0$ constant. Thus (84) is a constant times the yield (76), and the corresponding optimization problem reduces to that already discussed. When the *age distribution is not initially persistent* this reduction does not take place; the resulting problem is treated in [11].

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This paper develops optimal harvesting policies for a population whose evolution is governed by a single autonomous nonlinear differential equation. The objective functional is not assumed to be convex. The results are used to discuss optimal policies for age-structured populations harvested with effort independent of age.

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